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Boltzmann Equations For Mixtures of Maxwell Gases: Exact Solutions and Power Like Tails

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We consider the Boltzmann equations for mixtures of Maxwell gases. It is shown that in certain limiting case the equations admit self-similar solutions that can be constructed in explicit form. More precisely, the solutions have simple explicit integral representations. The most interesting solutions have finite energy and power like tails. This shows that power like tails can appear not just for granular particles (Maxwell models are far from reality in this case), but also in the system of particles interacting in accordance with laws of classical mechanics. In addition, non-existence of positive self-similar solutions with finite moments of any order is proven for a wide class of Maxwell models.

KEY WORDS: Boltzmann equation gas mixture Maxwell models exact solutions high energy tails

To Carlo Cercignani, on his 65th Birthday

1. INTRODUCTION

In this paper we continue the study of self-similar solutions for various physical systems described by the Boltzmann equations with Maxwell collision kernels.[(1–4)] It was understood long ago[(5)] that the key mathematical tool for such equations is the Fourier transform in the velocity space. A detailed analytical theory of the classical spatially homogeneous Boltzmann equation for Maxwell molecules was mainly completed in the 80's (see[(6)] for a review). There were almost no new essential results in this field during the 90's, except for some interesting pure mathematical publications, in particular.[(7,8)]

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Quite unexpectedly, Maxwell models again became the subject of many publications in the beginning of the 2000's. The starting point was an idea to use such models for descriptions of inelastic (granular) gases. Inelastic Maxwell models were introduced in 2000[1] (see also[9] for the one dimensional case). It was clear that all the analytical techniques previously developed for classical Maxwell models can be used in the inelastical case almost without changes. Many references to papers published in 2000's by physicists can be found in the book.[10]

One interesting result (absent in the elastic case) was the appearance of self-similar solutions with power like tails. It was conjectured in[11] and later proved in,[3,4] that such solutions represent asymptotic states for a wide class of initial data. On the other hand, inelastic Maxwell models are just a rough approximation for the inelastic hard sphere model and they give usually wrong answers to the question of large velocity asymptotics (a mathematically rigorous study of such asymptotics for a hard sphere model can be found in[12]).

Some new results in the theory of classical (elastic) Boltzmann equation for Maxwell molecules were also recently published in:[2,3] Self-similar solutions (two of which were found in explicit form) and the proof that such solutions represent a large time asymptotics for initial data with infinite energy, clarification of the old Krook-Wu conjecture,[13] etc. It is clear that both elastic and inelastic Maxwell models must be studied from a unified point of view.

An interesting question arises in connection with power-like tails for high velocities: Is it possible to observe a similar effect (an appearance of power-like tails from initial data with exponential tails) in the system of particles interacting according to laws of classical mechanics (i.e. without inelasticity assumption)? This is the main question for the present paper.

We shall see below that the answer to this question is probably affirmative. The key idea is to consider a mixture of classical Maxwell gases and to find a corresponding limiting case for which such behavior can be in principle observed.

The paper is organized as follows. First we consider the Boltzmann equation for Maxwell mixtures and pass to the Fourier representation (Sec 2). Then we study a binary mixture and show that corresponding equations formally admit a class of self-similar solutions (Sec 3). In order to simplify the problem we pass to the limit that corresponds to a one component gas in the presence of the thermostat with fixed temperature T (Sec 4). The general problem can be reduced to the case $T = 0$ (cold thermostat). Then we consider the case of the model cross section (pseudo-Maxwell molecules with isotropic scattering) and construct a family of exact self-similar solutions (Sec 5). These solutions are studied in detail in Sections 6 and 7. The new solutions have a lot in common with exact solutions from.[2] They, however, have finite energy and therefore are more interesting for applications.

We did not try to prove neither existence of such solutions in the more general case nor to show that they are large time asymptotic states for a wide class of initial data. This is done, as a particular case, in our paper[14] jointly with C. Cercignani.

Instead, we prove in Section 8 a general statement (applied, in particular, to inelastic Maxwell models and elastic models in the cold thermostat) concerning non-existence of positive self-similar solutions with finite moments of any order.

Thus it is sufficient in many cases to prove that such positive solution does exist, then it always has just a finite number of even integer moments for all values of parameters of the equation.

2. MAXWELL MIXTURES

We consider a spatially homogeneous mixture of $N \geq 2$ Maxwell gases. Each component of the mixture is characterized by the molecular mass m_i and the distribution function $f_i = f_i(v, t), i = 1, \ldots, N$, where $v \in \mathbb{R}^3$ and $t \in \mathbb{R}_+$ denote velocity and time respectively. The distribution functions are normalized in such a way that

$$\int_{\mathbb{R}^3} dv f_i(v, t) = \rho_i$$

where ρ_i is the number density of the i^{th} component of the mixture. Note that the quantities $\rho_i, i = 1, \ldots, N$ are preserved in time.

The interaction between particles is described by the matrix of Maxwell type differential cross-sections

$$\sigma_{ij}(|u|, \theta) = \frac{1}{|u|} g_{ij}(\cos\theta), \qquad i, j = 1, \ldots, N.$$

where $|u|$ is the relative speed of colliding particles, $\theta \in [0, \pi]$ is the scattering angle.

In case of “true” Maxwell molecules, i.e. particles interacting with potentials

$$U_{ij} = \frac{\alpha_{ij}}{r^4}, \qquad \alpha_{ij} > 0,$$

where $r > 0$ denotes a distance between interacting particles, the following formulas are valid[15]

$$g_{ij}(\cos\theta) = \left(\frac{\alpha_{ij}}{m_{ij}}\right)^{1/2} g(\cos\theta), \qquad m_{ij} = \frac{m_i m_j}{m_i + m_j}, \quad i, j = 1, \ldots, N.$$

We shall assume below the same kind of formulas for $g_{ij}(\cos\theta)$ with an arbitrary function $g(\cos\theta)$ (pseudo-Maxwell particles).

The corresponding system of Boltzmann equations reads

$$\frac{\partial f_i}{\partial t} = \sum_{j=1}^{N} \int_{\mathbb{R}^3 \times S^2} dv_* \, d\omega \, g_{ij}\left(\frac{u \cdot \omega}{|u|}\right) [f_i(v') f_j(v'_*) - f_i(v) f_j(v_*)] \qquad (2.1)$$

where the pair (v', v'_*) are pre-collisional velocities

$$v' = (m_i v + m_j v_* + m_j |u|\omega)(m_i + m_j)^{-1},$$
$$v'_* = (m_i v + m_j v_* - m_i |u|\omega)(m_i + m_j)^{-1}; \quad i, j = 1, \ldots, N \tag{2.2}$$

with respect to the post-collisional velocities (v, v_*).

The Fourier transform of Eqs. (2.1)–(2.2), for

$$\varphi_j(k, t) = \int_{\mathbb{R}^3} dv\ f_j(v, t)\, e^{-ik\cdot v}; \quad k \in \mathbb{R}^3$$

leads to equations[6]

$$\frac{\partial \varphi_i}{\partial t} = \sum_{J=1}^{N} \mathcal{S}(\varphi_i, \varphi_j) \tag{2.3}$$

where

$$\mathcal{S}(\varphi_i, \varphi_j) = \int_{S^2} d\omega\, g_{ij}\left(\frac{k\cdot\omega}{|k|}\right)[\varphi_i(k_{ij}^+, t)\,\varphi_j\,(k_{ij}^-, t) - \varphi_i(k, t)\,\varphi_j\,(0, t)] \tag{2.4}$$

where

$$k_{ij}^+ = \frac{m_i k + m_j |k|\omega}{m_i + m_j}, \qquad k_{ij}^- = \frac{m_j}{m_i + m_j}(k - |k|\omega)$$

so,

$$k = k_{ij}^+ + k_{ij}^-; \quad i, j = 1, \ldots, N.$$

We note that

$$(k_{ij}^-)^2 = 4\left(\frac{m_j}{m_i + m_j}\right)^2 |k|^2 s, \qquad |k_{ij}^+|^2 = |k|^2\left(1 - 4\frac{m_i m_j}{(m_i + m_j)^2} s\right),$$

where

$$s = \frac{1}{2}\left(1 - \frac{k\cdot\omega}{|k|}\right), \qquad s \leq 1.$$

Then, we consider isotropic solutions of Eqs. (2.3)–(2.4)

$$\varphi_i(k, t) = \tilde{\varphi}_i\left(\frac{|k|^2}{2m_i}, t\right),$$

and obtain

$$\frac{\partial}{\partial t}\tilde{\varphi}_i\left(\frac{|k|^2}{2m_i}, t\right) = \sum_{j=1}^{N} \tilde{\mathcal{S}}(\tilde{\varphi}_i, \tilde{\varphi}_j), \tag{2.5}$$

with

$$\tilde{S}(\tilde{\varphi}_i, \tilde{\varphi}_j) = \int_0^1 ds G_{ij}(s) \left[\tilde{\varphi}_i \left(\frac{|k|^2}{2m_i}(1 - \beta_{ij}s), t \right) \right.$$
$$\left. \tilde{\varphi}_j \left(\frac{|k|^2}{2m_j} \beta_{ij} \frac{m_j}{m_i} s, t \right) - \tilde{\varphi}_i \left(\frac{|k|^2}{2m_i}, t \right) \varphi(0, t) \right]$$

and

$$\beta_{ij} = \frac{4m_i m_j}{(m_i + m_j)^2}, \qquad G_{ij} = 4\pi g_{ij}(1 - 2s)$$

for $i, j = 1, \ldots, N$. We omit the tildes and denote

$$x = \frac{|k|^2}{2m_i}$$

in each of Eq. (2.5). Then the resulting set of equations becomes

$$\frac{\partial}{\partial t}\varphi_i(x, t) = \sum_{j=1}^{N} \int_0^1 ds G_{ij}(s)[\varphi_i(x(1 - \beta_{ij}s))\varphi_j(x\beta_{ij}s) - \varphi_i(x)\varphi_j(0)]$$

where

$$0 \le \beta_{ij} = 4\frac{m_i m_j}{(m_i + m_j)^2} \le 1, \qquad \beta_{ij} = \beta_{ji}, \quad \beta_{ii} = 1, \quad i, j = 1, \ldots, N.$$

Therefore, the most general system of isotropic Fourier transformed Boltzmann equations for Maxwell mixtures reads

$$\frac{\partial \varphi_i}{\partial t} = \sum_{j=1}^{N} \gamma_{ij} \langle \varphi_i((1 - \beta_{ij}s)x)\varphi_j(\beta_{ij}s\, x) - \varphi_i(x)\varphi_j(0) \rangle, \tag{2.6}$$

where, for any function $A(s)$, $s \in [0, 1]$,

$$\langle A(s) \rangle = \int_0^1 A(s)G(s)ds, \qquad G(s) = 4\pi g(1 - 2s),$$

and

$$\gamma_{ij} = \sqrt{\frac{\alpha_{ij}}{m_{ij}}}, \quad \beta_{ij} = \frac{4m_{ij}^2}{m_i m_j}, \quad m_{ij} = \frac{m_i m_j}{m_i + m_j};\ i, j = 1, \ldots, N. \tag{2.7}$$

3. BINARY MIXTURE

We consider a special case $N = 2$ in Eqs. (2.6), (2.7) and denote

$$m_1 = M,\ m_2 = m, \qquad \varphi_1 = \rho_1 \varphi(x, t), \quad \varphi_2 = \rho_2 \psi(x, t),$$

such that $\varphi(0,t) = \psi(0,t) = 1$. Then, we obtain

$$\varphi_t = \rho_1 \left(\frac{2\alpha_{11}}{M}\right)^{1/2} \langle \varphi, \varphi \rangle + \rho_2 \left(\frac{\alpha_{12}}{m_{12}}\right)^{1/2} \langle \varphi, \psi \rangle_\beta$$

$$\psi_t = \rho_2 \left(\frac{2\alpha_{22}}{m}\right)^{1/2} \langle \psi, \psi \rangle + \rho_1 \left(\frac{\alpha_{12}}{m_{12}}\right)^{1/2} \langle \psi, \varphi \rangle_\beta \tag{3.1}$$

where

$$\beta = \frac{4mM}{(m+M)^2}, \qquad m_{12} = \frac{mM}{m+M},$$

$$\langle \varphi, \psi \rangle_\beta = \int_0^1 ds\, G(s)\{\varphi((1-\beta s)x, t)\psi(\beta s x, t) - \varphi(x,t)\psi(0,t)\}$$

following the notation of (2.7), and thus,

$$\langle \varphi, \varphi \rangle = \langle \varphi, \varphi \rangle_1, \qquad \langle \psi, \psi \rangle = \langle \psi, \psi \rangle_1.$$

We recall the connection of functions $\varphi(x,t)$ and $\psi(x,t)$ with corresponding solutions $f_{1,2}(|v|,t)$ of the Boltzmann Eq. (2.1)

$$\rho_1 \varphi\left(\frac{|k|^2}{2M}, t\right) = \int_{\mathbb{R}^3} dv f_1(|v|,t) e^{-ik\cdot v},$$

$$\rho_2 \psi\left(\frac{|k|^2}{2m}, t\right) = \int_{\mathbb{R}^3} dv f_2(|v|,t) e^{-ik\cdot v}$$

The usual definition of kinetic temperatures is given by equalities

$$T_i = \frac{m_i}{3\rho_i} \int_{\mathbb{R}^3} dv |v|^2 f_i(|v|,t), \quad i = 1, 2.$$

Then, one can easily verify that

$$T_1(t) = -\varphi'(0,t), \qquad T_2(t) = -\psi'(0,t).$$

where the primes denote derivatives on x. The equilibrium temperatures T_{eq} of the binary mixture reads

$$T_{eq} = \frac{\rho_1 T_1 + \rho_2 T_2}{\rho_1 + \rho_2} = \text{const.}$$

The relaxation process in the binary mixture described by Eq. (3.1) leads to usual Maxwell asymptotics states

$$\varphi \to_{t\to\infty} \exp(-T_{eq}x), \qquad \psi \to_{t\to\infty} \exp(-T_{eq}x).$$

By using Eq. (3.1), one can easily verify (at the formal level) that

$$\frac{dT_1}{dt} = -\lambda\rho_2(T_1 - T_2), \quad \frac{dT_2}{dt} = -\lambda\rho_1(T_2 - T_1),$$
$$\lambda = \left(\frac{\alpha_{12}}{m_{12}}\right)^{1/2} \beta\langle s\rangle, \quad \langle s\rangle = \int_0^1 ds\, G(s)s.$$

Therefore,

$$T_1(t) = T_{eq} + \frac{\Delta}{\rho_1}e^{-\Lambda t},\ T_2(t) = T_{eq} + \frac{\Delta}{\rho_2}e^{-\Lambda t},$$
$$\Delta = \frac{\rho_1\rho_2}{\rho_1+\rho_2}(T_1(0) - T_2(0)),\ \Lambda = \lambda(\rho_1 + \rho_2). \tag{3.2}$$

It is easy to see that Eq. (3.1) formally admit the following class of self-similar solutions

$$\varphi(x,t) = \Phi(x\,e^{-\Lambda t})\,e^{-T_{eq}x}, \qquad \psi(x,t) = \Psi(x\,e^{-\Lambda t})e^{-T_{eq}x}$$

It is, however, difficult to investigate such solutions (in particular, to prove that corresponding distribution functions are positive) in the most general case. Therefore we consider a simplified problem.

4. WEAKLY COUPLED BINARY MIXTURE

If the masses M and m are fixed, then Eq. (3.1) contain five positive parameters $p_i, \alpha_{ij}, j = 1, 2$. We shall consider below a special limiting case of Eq. (3.1) (weakly interacting gases) such that

$$\alpha_{12} \to 0, \quad \rho_2 \to \infty, \quad \rho_2\sqrt{\alpha_{12}} = \text{const.} \tag{4.1}$$

We assume that the other parameters ρ_1, α_{11} and α_{22} remain constant and denote

$$\varphi(x,t) = \tilde{\varphi}(x,\tilde{t}), \psi(x,t) = \tilde{\psi}(x,\tilde{t}), \qquad \tilde{t} = \rho_1\left(\frac{2\alpha_{11}}{M}\right)^{1/2} t, \tag{4.2}$$
$$\theta = \frac{\rho_2}{\rho_1}\left(\frac{\alpha_{12}M}{2\alpha_{11}m_{12}}\right)^{1/2} = \text{const.}$$

Then we formally obtain (tildes are omitted below)

$$\varphi_t = \langle\varphi,\varphi\rangle + \theta\langle\varphi,\psi\rangle_\beta, \qquad \langle\psi,\psi\rangle = 0, \tag{4.3}$$

where it is assumed that the functions $\varphi(x,t)$, $\psi(x,t)$ and their time derivatives remain finite in the limit (4.1). The limiting temperatures (see Eq. (3.2)) are given by the identities

$$T_2(t) = T_2(0) = T_{eq} = \text{const.}, \quad \text{and} \quad T_1(t) = T_2(0) + (T_1(0) - T_2(0))e^{-\theta\beta\langle s\rangle t}$$

in the notation (4.2) (tildes are omitted).

Thus

$$< \psi, \psi >= 0, \ \psi(0,t) = 1, \ \psi'(0,t) = -T_2(0) \Rightarrow \psi(x,t) = e^{-T_2(0)x},$$

and we reduce Eq. (4.3) to the unique equation for $\varphi(x,t)$

$$\frac{\partial\varphi}{\partial t} = \langle\varphi,\varphi\rangle + \theta\langle\varphi, e^{-T_2(0)x}\rangle_\beta. \tag{4.4}$$

The general case $T_2(0) > 0$ can be reduced to the case $T_2(0) = 0$ by substitution

$$\varphi(x,t) = \approx{\varphi}(x,t)\exp(-T_2(0)x).$$

Eq. (4.4) shows that, in the limiting case (4.1), the second component of the mixture plays a role of a thermostat with the fixed temperature $T_2(0)$, moreover, it is enough to consider the case $T_2(0) = 0$ (cold thermostat). Then Eq. (4.4), in explicit form, reads

$$\frac{\partial\varphi}{\partial t} = \int_0^1 ds G(s)\{\varphi(sx)\varphi[(1-s)x] + \theta\varphi[(1-\beta s)x] - \varphi(x)[\varphi(0) + \theta\psi(0)]\},$$
$$\varphi(0) = \psi(0) = 1, \tag{4.5}$$

where the argument t of $\varphi(x,t)$ is omitted.

We consider below Eq. (4.5) assuming that $\varphi(|k|^2, 0)$ is a characteristic function (Fourier transform of a probability measure in $\mathbb{R}^3$). Then Eq. (4.5) describes a homogeneous cooling process in the system of particles that interact between themselves and with the cold thermostat. Though all interactions are elastic (at the microlevel), such system has much in common with the gas of inelastic particles considered in.[3,4] It was proved in these papers that the general inelastic Maxwell model has the self-similar asymptotics in a certain precise sense. We conjecture the same asymptotic property for solutions of Eq. (4.5) (see[14] for its proof). In this paper we construct some explicit examples of self-similar solutions and show that such solutions have power-like tails for large velocities.

5. EXACT SOLUTIONS IN THE FOURIER-LAPLACE REPRESENTATION

We consider Eq. (4.5) with $\beta = 1$ (equal masses $m = M = 1$) and $G(s) = 1$ (isotropic scattering). Then Eq. (4.5) reads

$$\frac{\partial\varphi}{\partial t} = \int_0^1 ds\varphi[(1-s)x][\varphi(sx) + \theta] - (1+\theta)\varphi(x), \quad \varphi(0) = 1. \tag{5.1}$$

From the physical point of view, this is a model for a mixture of two weakly interacting gases, which consist of "particles" with identical masses. One can assume that the sorts of "particles" differ, say, by color.

Our goal is to describe a family of self-similar solutions of Eq. (5.1) such that

$$\varphi(x,t) = \psi(xe^{-\mu t}), \quad \psi(x) \cong 1 - ax^p, \quad x \to 0, \tag{5.2}$$

The parameters $\theta \geq 0$, $p > 0$ and $\mu \in \mathbb{R}$ will be determined later. From now on x denotes the self-similar variable. (The notation $\psi(xe^{-\mu t})$ should not be confused with one for the function $\psi(x,t)$ from Sections 3 and 4.)

Substituting Eq. (5.2) into Eq. (5.1) we obtain

$$\mu x\psi'(x) - (1+\theta)\psi(x) + \frac{1}{x}\psi * (\psi + \theta) = 0,$$

$$\psi_1 * \psi_2 = \int_0^x dy \psi_1(y)\psi_2(x-y). \tag{5.3}$$

This equation can be simplified by the use of the Laplace transform similarly to.[2]

$$w(z) = \mathcal{L}(\psi)(z) = \int_0^\infty \psi(x)e^{-zx}, \quad \operatorname{Re} z > z_0, \tag{5.4}$$

provided $|\psi(x)| < A\exp(z_0 x)$, with some positive A and z_0.

First, we recall properties of the Laplace transform

$$\mathcal{L}(x\psi) = -w'(z), \quad \mathcal{L}(x^2\psi) = w''(z) \quad \text{and} \quad \mathcal{L}(\psi') = zw(z) - \psi(0),$$

and

$$\mathcal{L}(x^2\psi') = \frac{d^2}{dz^2}(zw - \psi(0)) = (zw(z))''.$$

Then we obtain the following equation for $w(z)$:

$$\mu(zw)'' + (1+\theta)w' + w\left(w + \frac{\theta}{z}\right) = 0.$$

Next, we denote

$$u(z) = zw(z) = \int_0^\infty dx\, e^{-x}\psi\left(\frac{x}{z}\right), \tag{5.5}$$

so that the above equation is transformed to

$$\mu z^2 u'' + (1+\theta)zu' + u(u-1) = 0. \tag{5.6}$$

The next step is to simplify this equation by standard substitutions. We denote

$$\tilde{z} = z^q, \quad u(z) = \tilde{u}(\tilde{z}), \tag{5.7}$$

and obtain the equation for $\tilde{u}(\tilde{z})$ (tildes are omitted)

$$\mu q^2 z^2 u'' + q[1 + \theta + \mu(q - 1)]zu' + u(u - 1) = 0.$$

Then, setting

$$u(z) = z^2 y(z) + B, \quad B = \text{const.}, \tag{5.8}$$

we obtain the following equation for $y(z)$:

$$\mu q^2 z^4 y'' + z^4 y^2 + \alpha z^3 y' + \beta z^2 y + B(B - 1) = 0, \tag{5.9}$$

where the parameters α and β are given by the relations

$$\alpha = q(5\mu q + 1 + \theta - \mu)\,,$$
$$\beta = 2B - 1 + 4\mu q^2 + 2q(1 + \theta - \mu)$$

Now, the parameters q and B can be chosen in such a way that $\alpha = \beta = 0$. Thus, we take

$$q = -\frac{1 + \theta - \mu}{5\mu}, \qquad B = \frac{6\mu q^2 + 1}{2},$$

and obtain that Eq. (5.9) is reduced to

$$\mu q^2 y'' + y^2 + \frac{B(B - 1)}{z^4} = 0. \tag{5.10}$$

This is the simplest standard form, for which Eq. (5.6) can be reduced in the general case. Finally, Eq. (5.10) is of the Painlevé type if and only if $B = 0$ or $B = 1$. Otherwise, it has moving logarithmic singularities and does not have any “simple” nontrivial analytic solutions (this is just a repetition of arguments on a similar equation considered in[(2)]).

Therefore we consider the two special cases. In both cases

$$\theta = \mu - 1 - 5\mu q. \tag{5.11}$$

Case 1: $B = 0$, which implies

$$6\mu q^2 = -1, \qquad \text{so that} \qquad y'' = 6y^2. \tag{5.12}$$

Case 2: $B = 1$, which implies

$$6\mu q^2 = -1, \qquad \text{so that} \qquad y'' = -6y^2. \tag{5.13}$$

The general solution of the non-linear ODE $y'' = by^2$ is expressed in terms of Weierstrass elliptic function. Similarly to,[(2)] we can show that just the simplest exact solutions, namely

$$\textbf{Case 1:} \quad y(z) = (c_1 + z)^{-2} \qquad \text{and} \qquad \textbf{Case 2:} \quad y(z) = -(c_2 + z)^{-2},$$

with constant c_1 and c_2, are solutions to Eqs. (5.12) and (5.13) respectively, that lead to solutions of Eq. (5.6) satisfying appropriate boundary conditions at infinity.

Coming back to the original notation (see the transformations (5.7), (5.8)), we obtain, therefore, two different exact solutions of Eq. (5.6)

$$\begin{aligned} &\textbf{Case 1: } u(z) = (1 + c_1 z^{-q})^{-2} \ \text{ for } \ 6\mu q^2 = -1; \\ &\textbf{Case 2: } u(z) = 1 - (1 + c_2 z^{-q})^{-2} \ \text{ for } \ 6\mu q^2 = -1. \end{aligned} \tag{5.14}$$

We have that, in both cases, the coupling constant θ must satisfy (5.11).

The constants c_1 and c_2 are determined by the boundary conditions as follows. The given asymptotics in (5.2) for $\psi(x)$, $x \to 0$, leads to the asymptotics for $u(z)$, as defined in (5.5), at infinity

$$u(z) \simeq 1 + \frac{b}{z^p}, \qquad z \to \infty \, ,$$

where b is a non-zero constant. Recalling that $|\psi(x)| \leq 1$ for positive solutions of the Boltzmann equation, we can assume, without loss of generality, that

$$u(z) \simeq 1 + \frac{1}{z^p}, \qquad z \to \infty \, . \tag{5.15}$$

Then we obtain, for the above two cases in (5.14), the following formulas satisfying the boundary condition (5.15) at infinity:

$$\begin{aligned} &\textbf{Case 1: } q = p; \quad u(z) = \left(1 + \frac{1}{2} z^{-p}\right)^{-2}, \quad \mu = -\frac{1}{6p^2}; \\ &\qquad \theta = \frac{(3p-1)(1-2p)}{2p^2}. \\ &\textbf{Case 2:} q = -\frac{p}{2}; \quad u(z) = 1 - (1 + z^{p/2})^{-2}, \quad \mu = \frac{2}{3p^2}; \\ &\qquad \theta = \frac{(3p+1)(2-1p)}{3p^2}. \end{aligned} \tag{5.16}$$

The result can be formulated in the following way.

Proposition 5.1. *Eq. (5.1) has exact self-similar solutions (5.2) satisfying the condition*

$$\psi(x) \simeq 1 - \frac{x^p}{\Gamma(p+1)}, \qquad x \to 0, \quad p > 0,$$

for the following values of the parameters $\theta(p)$ and $\mu(p)$:

$$\mathbf{1}: \mu(p) = -\frac{1}{6p^2}, \quad \theta(p) = \frac{(3p-1)(1-2p)}{6p^2};$$
$$\mathbf{2}: \mu(p) = \frac{2}{3p^2}, \quad \theta(p) = \frac{(3p+1)(2-p)}{3p^2}; \tag{5.17}$$

The solutions of Eq. (5.1) are given by equalities

$$\psi_i(x) = \mathcal{L}^{-1}\left[\frac{u_i(z)}{z}\right], \qquad i = 1, 2, \tag{5.18}$$

with $u_{1,2}(z)$ from Eq. (5.16), for cases 1 and 2 respectively.

The solutions have a physical meaning if, both, $\theta \geq 0$ and $\psi\left(\frac{|k|^2}{2}\right)$ is the Fourier transform of a positive function (measure).

The first condition leads to inequalities $\frac{1}{3} \leq p \leq \frac{1}{2}$ in the case 1, and to $0 \leq p \leq 2$ in the case 2. The second condition will be discussed in the next section.

6. DISTRIBUTION FUNCTIONS

First we evaluate the inverse Laplace transforms (5.18). In the case 1 we obtain

$$\psi_i(x) = \mathcal{L}^{-1}\left[\frac{1}{z}\left(1 + \frac{z^{-p}}{2}\right)^2\right], \qquad \frac{1}{3} \leq p \leq \frac{1}{2}.$$

The general formula from[17] leads to

$$\psi_1(x) = \Phi(2^{-1/p}x), \qquad \Phi(x) = \mathcal{L}^{-1}\left[\frac{1}{z}(1+z^{-p})^{-2}\right]$$
$$= 2\frac{\sin p\pi}{p\pi}\int_0^\infty ds\, e^{-xs^{-1/p}}\frac{(1+s\cos p\pi)}{(1+s^2+2s\cos p\pi)^2}. \tag{6.1}$$

This case corresponds to

$$\theta = \frac{(3p-1)(1-2p)}{6p^2} \tag{6.2}$$

in Eq. (5.1). We note that $\theta = 0$ for $p = \frac{1}{3}, \frac{1}{2}$. Eq. (5.1) in such cases is the Fourier transformed Boltzmann equation for one-component gas. The exact solutions (6.1) with $p = \frac{1}{2}, \frac{1}{3}$ were already obtained in.[2] Eq. (6.1) therefore yields a generalization of these solutions to the case of binary mixture (with equal masses) provided the parameter θ is given (for given $p \in [1/3, 1/2]$) in Eq. (6.2). The corresponding solutions of the Boltzmann equations are positive solutions with infinite energy. All their properties can be studied in the same way as in.[2]

The case 2 is more interesting since it includes also solutions with finite energy. The inverse Laplace transform

$$\psi_2(x) = \mathcal{L}^{-1}\left\{\frac{1}{z}\left[1 - \frac{1}{(1+z^{p/2})^2}\right]\right\}$$

can be evaluated in the following way. We denote

$$\Psi(x) = \mathcal{L}^{-1}\left[\frac{1}{(1+z^{p/2})^2}\right],$$

then

$$\psi_2(x) = 1 - \int_0^x dy \Psi(y) = \int_x^\infty dy \Psi(y), \quad \int_0^\infty dy \Psi(y) = 1. \tag{6.3}$$

The function $\Psi(x)$ can be expressed through the integral

$$\Psi(x) = \frac{1}{2\pi i}\int_C \frac{dz\, e^{xz}}{(1+z^{p/2})^2}, \qquad 0 < p \le 2$$

where the contour C lies around the negative half of the real axis (see[17] for details).

Then we obtain

$$\Psi(x) = \frac{1}{\pi}\int_0^\infty dr\, e^{-rx}\, A(r),$$

where

$$A(x) = \mathrm{Im}\,[1 + (re^{-i\pi})^{p/2}]^{-2} = \frac{2r^{p/2}(1 + r^{p/2}\cos\frac{p\pi}{2})\sin\frac{p\pi}{2}}{(1 + r^p + 2r^{p/2}\cos\frac{p\pi}{2})^2}$$

Coming back to the function $\psi_2(x)$ (6.3), we obtain

$$\psi_2(x) = \frac{1}{\pi}\int_0^\infty \frac{dr\, e^{-rx} A(r)}{r}. \tag{6.4}$$

The final result

$$\psi_2(x) = \frac{4\sin\frac{p\pi}{2}}{p\pi}\int_0^\infty \frac{ds(1 + s\cos\frac{p\pi}{2})}{(1 + s^2 + 2s\cos\frac{p\pi}{2})^2} e^{-xs^{2/p}}, \quad 0 < p \le 2,$$

is obtained by substitution $s = r^{p/2}$ in the integral (6.4).

We remind to the reader that the corresponding distribution function $f(|v|, t)$, that solves the Boltzmann equation, reads

$$f(|v|, t) = e^{3\mu t/2} F(|v| e^{\mu t/2}), \qquad \mu = \frac{2}{3p^2}, \tag{6.5}$$

where

$$\mathcal{F}[F] = \int_{\mathbb{R}^3} dv F(|v|) e^{-ik\cdot v} = \psi_2\left(\frac{|k|^2}{2}\right).$$

Noting that

$$e^{-T\frac{|k|^2}{2}} = \mathcal{F}[M_T(|v|)], \qquad M_T(|v|) = \frac{e^{-\frac{|v|^2}{2T}}}{(2\pi T)^{3/2}},$$

we obtain the integral representation of $F(|v|)$:

$$F(|v|) = \frac{4\sin\frac{p\pi}{2}}{p\pi}\int_0^\infty ds \frac{(1 + s\cos\frac{p\pi}{2})}{(1 + s^2 + 2s\cos\frac{p\pi}{2})^2} M_{s^{2/p}}(|v|). \tag{6.6}$$

This function is obviously positive for $0 < p \leq 1$. We note that the solution (6.5)–(6.6) corresponds to the value

$$\theta = \frac{(3p + 1)(2 - p)}{3p^2}$$

in Eq. (5.1).

7. SOLUTIONS WITH FINITE ENERGY

We consider in more detail the most interesting (for applications) case $p = 1$ in Eq. (6.6). Then

$$\mu = \frac{2}{3}, \quad \theta = \frac{4}{3}, \qquad f(|v|, t) = e^t F(|v| e^{t/3}), \tag{7.1}$$

where

$$F(|v|) = \frac{4}{\pi}\int_0^\infty ds \frac{\exp(-|v|^2/2s^2)}{(2\pi s^2)^{3/2}(1 + s^2)^2}. \tag{7.2}$$

We denote

$$y = \frac{|v|^2}{2}, \qquad F(|v|) = (2\pi^5)^{-1/2}\Phi(y), \tag{7.3}$$

where

$$\Phi(y) = \int_0^\infty dr \frac{r^2}{(1 + r)^2} e^{-ry} = \frac{1}{y^3}\int_0^\infty dr \frac{r^2 e^{-r}}{(1 + \frac{r}{y})^2}. \tag{7.4}$$

Asymptotic expansion of $\Phi(y)$ for large positive y follows from integration of the formal series

$$\left(1 + \frac{r}{y}\right)^{-2} = \sum_{n=0}^{\infty}(-1)^n (n + 1)\left(\frac{r}{y}\right)^n.$$

Thus we obtain from Eq. (7.4)

$$\Phi(y) \simeq \frac{1}{y^3} \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)!}{y^n}, \qquad y \to \infty \tag{7.5}$$

In order to describe a behavior of $\Phi(y)$ for small positive y we transform the first integral in Eq. (7.4) in the following way

$$\Phi(y) = \int_1^{\infty} dr \frac{(r-1)^2}{r^2} e^{-(r-1)y} = e^y [E_0(y) - 2E_1(y) + E_2(y)],$$

$$E_m(y) = \int_1^{\infty} dr \frac{e^{-ry}}{r^m}.$$

Noting that

$$E_0(y) = \frac{e^{-y}}{y}, \quad E_2(y) = e^{-y} - yE_1(y),$$

we obtain

$$\Phi(y) = 1 + \frac{1}{y} - (2+y)e^y E_1(y),$$

where

$$E_1(y) = \int_y^{\infty} ds \frac{e^{-s}}{s} = \int_1^{\infty} ds \frac{e^{-s}}{s} + \int_y^1 ds \frac{(e^{-s}-1)}{s} - \ln y.$$

We note that

$$\int_1^{\infty} ds \frac{e^{-s}}{s} + \int_0^1 ds \frac{(e^{-s}-1)}{s} = \int_0^{\infty} ds\, e^{-s} \ln s = -\gamma,$$

where $\gamma \simeq 0, 577$ is the Euler constant. Therefore

$$E_1(y) = -(\gamma + \ln y) + \int_0^y ds \frac{(1-e^{-s})}{s}$$

and

$$\Phi(y) = 1 + \frac{1}{y} + (2+y)\left[e^y(\gamma + \ln y) - \int_0^y ds \frac{(e^s-1)}{s}\right]. \tag{7.6}$$

Thus, the asymptotic equality (7.5) and the formula (7.6) describe the behavior of the distribution function $F(v)$ in (7.3), for large and small values of $|v|$. We obtain

$$F(|v|) = 2\left(\frac{2}{\pi}\right)^{5/2} \frac{1}{|v|^6}\left[1 + O\left(\frac{1}{|v|}\right)\right], \qquad |v| \to \infty,$$

$$F(|v|) = \frac{2^{1/2}}{\pi^{5/2}} \frac{1}{|v|^2}[1 + 2|v|^2 \ln |v| + O(|v|^2)], \quad |v| \to 0. \tag{7.7}$$

All the exact solutions can be generalized to the case of the thermostat with finite temperature T (see Eq. (4.4) with $T_2(0) = T$). Then Eq. (5.1) is replaced by the following equation

$$\frac{\partial \varphi}{\partial t} = \int_0^1 ds\{\varphi(sx)\varphi[(1-s)x] - \varphi(x)\varphi(0)\}$$
$$+\theta \int_0^1 ds\{\varphi[(1-s)x]e^{-Tsx} - \varphi(x)\},$$
$$\varphi(0) = 1.$$

This equation can be reduced to Eq. (5.1) by substitution

$$\varphi(x,t) = \hat{\varphi}(x,t)e^{-Tx}.$$

The corresponding self-similar solutions read

$$\varphi(x,t) = \psi(xe^{-\mu t})e^{-Tx},$$

where $\psi(x)$ satisfies Eq. (5.3).

8. SELF-SIMILAR SOLUTIONS AND POWER LIKE TAILS

We consider in this section a more general class of equations for the function $\varphi(x,t)$:

$$\frac{\partial \varphi}{\partial t} = \int_0^1 ds G(s)\{\varphi(a(s)x)\varphi[b(s)x] - \varphi(x)\varphi(0)\}$$
$$+\theta \int_0^1 ds H(s)\{\varphi[c(s)x] - \varphi(x)\},$$
$$\varphi(0) = 1, \tag{8.1}$$

with non-negative functions $G(s)$, $H(s)$, $a(s)$, $b(s)$ and $c(s)$ with $s \in [0, 1]$. We also assume that $G(s)$, $H(s)$ are integrable on [0, 1], and

$$a(s) \leq 1, \ b(s) \leq 1, \ c(s) \ \leq 1; \ 0 \leq s \leq 1.$$

The function $\varphi(x,t)$ is understood as the Fourier transform

$$\varphi_(|k|^2, t) = \int_{\mathbb{R}^d} dv f(|v|,t)\, e^{-ik\cdot v}; \quad k \in \mathbb{R}^d, \ \ d = 1, 2, \ldots, \tag{8.2}$$

of a time dependent probability density $f(|v|,t) \in \mathbb{R}^d$.

Eq. (8.1) allows to consider from a unified point of view two different kind of Maxwell models:

I Inelastic Maxwell models,[1,3] where

$$H(s) = 0, \quad a(s) = sz^2, \quad b(s) = 1 - sz(2 - z),$$
$$z = \frac{1 + e}{2}, \quad 0 \le e \le 1. \tag{8.3}$$

II Maxwell mixtures described by Eq. (4.5), where

$$H(s) = \theta G(s), \quad a(s) = s, \quad b(s) = 1 - s, \quad c(s) = 1 - \beta s. \tag{8.4}$$

The condition of integrability of $G(s)$ and $H(s)$ can be easily weakened. We do not do it here in order to simplify proofs. Our main goal in this section is to prove, roughly speaking, that self-similar solutions (distribution functions $f(|v|, t)$ from Eq. (8.2)) have power-like tails. More precisely, we are going to show that such distribution functions can not have finite moments of any order.

Eq. (8.1) admits (formally) a class of self-similar solutions

$$\varphi(x, t) = \psi(xe^{-\mu t}), \quad \mu > 0, \tag{8.5}$$

where $\varphi(x, t)$ satisfies the equation

$$\begin{aligned} -\mu x\psi' &= \int_0^1 ds G(s)[\psi(a(s)x)\psi((b(s)x) - \psi(x)] \\ &\quad + \int_0^1 ds H(s)[\psi(c(s)x) - \psi(x)], \\ \psi(0) &= 1, \qquad \mu > 0. \end{aligned} \tag{8.6}$$

The corresponding function $f(|v|, t) \in \mathbb{R}^d$ (see Eq. (8.2)) reads

$$f(|v|, t) = e^{d\frac{\mu t}{2}} F(|v|e^{\frac{\mu t}{2}}), \qquad \psi(|k|^2) = \int_{\mathbb{R}^d} dv F(|v|)e^{-ik\cdot v}. \tag{8.7}$$

Our goal is to prove the following general fact: if such a function $F(|v|) \ge 0$ (generalized density of a probability measure in $\mathbb{R}^d$) does exists, then it can not have finite moments of all orders. We assume the opposite and represent the integral in Eq. (8.1) as a formal series

$$\psi(|k|^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \alpha_n(d) m_n |k|^{2n},$$

where

$$m_n = \int_{\mathbb{R}^d} dv\ F(|v|)|v|^{2n}, \quad n = 0, 1, \ldots;$$

$$\alpha_n(1) = 1, \quad \alpha_n(d) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} d\omega(\omega' \cdot \omega)^{2n}, \quad \text{if } d \geq 2.$$

where $|S^{d-1}|$ is the usual measure of the unit sphere S^{d-1} in $\mathbb{R}^d$, $\omega' \in S^{d-1}$ is an arbitrary unit vector.

Hence we obtain

$$\psi(|k|^2) = \sum_{n=0}^{\infty} (-1)^n \psi_n \frac{|k|^{2n}}{n!},$$

$$\psi_0 = 1, \ \psi_n = \frac{n!}{(2n!)} \alpha_n(d)\, m_n, \quad n = 1, 2, \ldots. \tag{8.8}$$

The convergence of the Taylor series (8.8) is irrelevant for our goals. The only important point is that $\psi(x)$ is infinitely differentiable for all $0 \leq x < \infty$ (see any textbook in probability theory, for example[16]) and

$$\psi^{(n)}(0) = \psi_n > 0, \quad n = 0, 1, \ldots. \tag{8.9}$$

On the other hand, the equations for ψ_n can be easily obtained by substitution of the series (8.8) into Eq. (8.6). Then

$$\psi_0 = 1, \quad \psi_1[\mu - \lambda(1)] = 0,$$

$$\psi_n[\mu n - \lambda(n)] + \sum_{k=1}^{n-1} \mathcal{G}(k, n-k)\psi_k \psi_{n-k} = 0, \quad n = 2, 3, \ldots,$$

where

$$\lambda(n) = \int_0^1 ds G(s)[1 - a^n(s) - b^n(s)] + \int_0^1 ds H(s)[1 - c^n(s)],$$

$$\mathcal{G}(k, l) = \binom{k+l}{k} \int_0^1 ds G(s) a^k(s) b^l(s), \qquad k, l = 1, 2, \ldots.$$

Now we can use conditions (8.9). First we obtain $\mu = \lambda(1)$ and recall that $\mu > 0$ by assumption (8.5). Then we note that $\mathcal{G}(k, l) \geq 0$ for all $k, l = 1, 2, \ldots$. Therefore

$$\psi_n[-\mu n + \lambda(n)] \geq 0 \implies \frac{\lambda(n)}{n} \geq \mu > 0, \quad n = 2, 3, \ldots.$$

On the other hand,

$$\lambda(n) \leq \int_0^1 ds[G(s) + H(s)] < \infty,$$

and, therefore, we get a contradiction.

Thus, the following statement has been proven.

Proposition 8.1. *Eq. (8.6), where* $\mu > 0, G(s), H(s) \in L_{+}[0, 1], 0 \leq a(s) \leq 1, 0 \leq b(s) \leq 1, 0 \leq c(s) \leq 1; s \in [0, 1]$*, does not have infinitely differentiable at* $x = 0$ *solutions satisfying conditions (8.9).*

Corollary 8.2. *The corresponding probability density* $F(|v|)$ *from (8.7) cannot have finite moments of all orders.*

We can now apply the result to the inelastic Maxwell models (8.3) and conclude that the similar statement proved in our first paper[1] on that subject (see,[1] Section 5, Theorem 5.1) is valid not just for almost all, but for all values of the restitution coefficient $e \in [0, 1]$. Consequently we can revise now a statement from[3] related to existence of self-similar solutions with finite moments of any order for a countable set of values of e from the interval set [0, 1] (a possible logarithmic singularity was missing in the sketch of proof of Theorem 7.2 in[3]). In fact the solution constructed in Ref. 3 has a finite number of moments for any $0 \leq e < 1$, without exceptions.

On the other hand, the above Proposition 8.1 can be applied to Maxwell mixtures (8.4). It shows that any *physical* (i.e. with a positive $F(|v|)$ in Eq. (8.7)) solution of Eq. (8.6) corresponds to the distribution function $F(|v|)$ with a finite number of even integer moments. The exact solution constructed in Section 5 can be considered as a typical example.

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